Summary

3. Laws of electromagnetism, Magnetic circuits, Flux linkage

An *Electrical Machine* is any reversible device able to convert electric energy into electric energy or into mechanical one (and vice-versa), based on the laws of electromagnetism.

H magnetic field intensity (*campo magnetico*) [A/m]

B magnetic flux density (*induzione magnetica*) [T]

Relationship between these two quantities

$$\overline{B} = \mu \overline{H}$$

where μ is the permeability (*permeabilità*) of the material.

For the air, the permeability is equal to the vacuum permeability $\mu_o = 4\pi\cdot\times10^{-7}$ [H/m]

The permeability of any material may be represented by a parameter (relative permeability μ_r) as the ratio between the permeability of the material (μ) and the permeability of the vacuum (μ_o):

$$\mu = \mu_r \mu_o$$

For a ferromagnetic material, the permeability is not constant, but it is a non-linear function of the magnetic field intensity **H**, as in Figure 3-1.

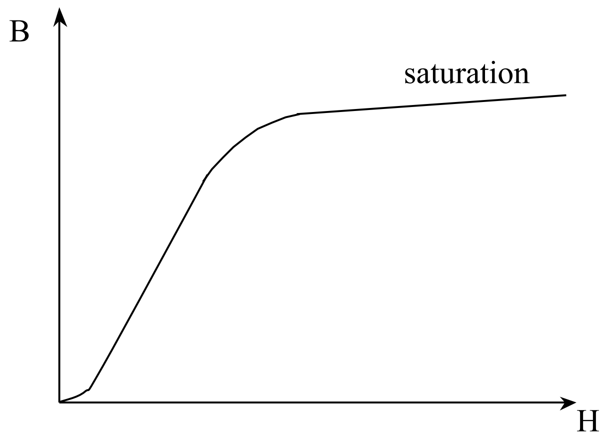

Figure 3-1: Initial magnetization curve (*caratteristica di prima magnetizzazione)*

Figure 3-1 represents the magnetization starting from a demagnetized condition. The slope of the characteristic in the saturation condition is very near to the vacuum permeability. The return path is not the same as before, so, in a sinusoidal condition, the trajectory describes a closed loop.

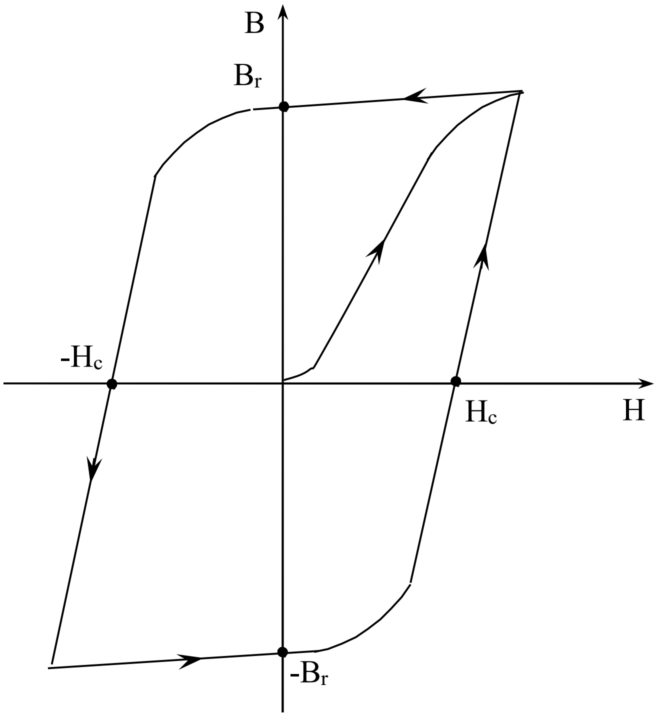

Figure 3-2: Magnetic Hysteresis curve (*ciclo di isteresi magnetica)*

H_c is the coercivity (*coercitività o campo coercitivo*) while B_r is Remanence or Remanent magnetization or Residual magnetism (*induzione residua*).

3.1 Ampere's Law

The line integral of magnetic field intensity H around a closed contour C is equal to the total current passing through any surface S linking (bounded by) that contour.

$$\oint_C \overline{H} \cdot d\bar{l} = \sum_k i_k$$

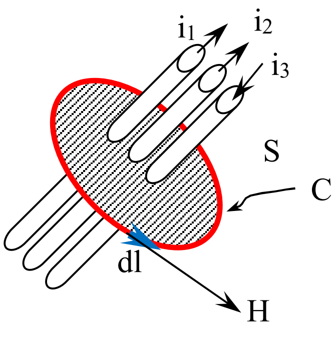

Figure 3-3: Graphical representation of the Ampere's law

The $\sum_k i_k$ is called Magnetomotive Force m.m.f. (M).

A flux line is defined as a locus of point for which the direction of the magnetic field (in that point) is tangent to the flux line

For a single wire in the air, carrying a current I, the symmetry suggests that the flux lines are circumferences whose center is the wire itself. The magnetic field intensity H (the same for the flux density B) is constant along that circumference (of radius *r*) and tangent to it (the versus may be found using the right hand rule).

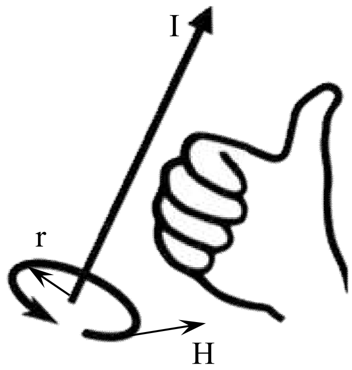

Figure 3-4: Right-hand law (*regola della mano destra*)

Therefore, the relationship may be reduced to:

$$\oint_C \overline{H} \cdot d\bar{l} = H 2\pi r = I$$

so

$$H = \frac{I}{2\pi r}$$

If we have N wires with the same current I:

$$H = \frac{NI}{2\pi r} = \frac{M}{2\pi r}$$

3.2 Gauss's Law and Magnetic flux definition

The surface integral of the normal component of the magnetic flux density B over a closed surface S is zero (it means that the flux lines are closed); the surface integral of the normal component of the magnetic flux density B over a non-closed surface S is the magnetic flux φ [Wb].

$$\oiint_S \overline{B} \cdot \bar{n} dA = 0$$

$$\iint_S \overline{B} \cdot \bar{n} dA = \varphi$$

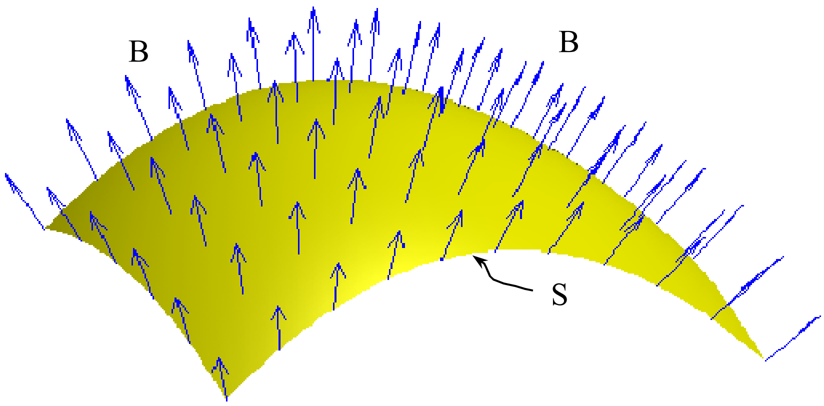

Figure 3-5: Graphical representation of the magnetic flux

Consider a surface S and its contour C; you may define the flux tube as the region of space bounded by the flux lines linked to each point of the contour C. Figure 3-6 shows a segment of a flux tube.

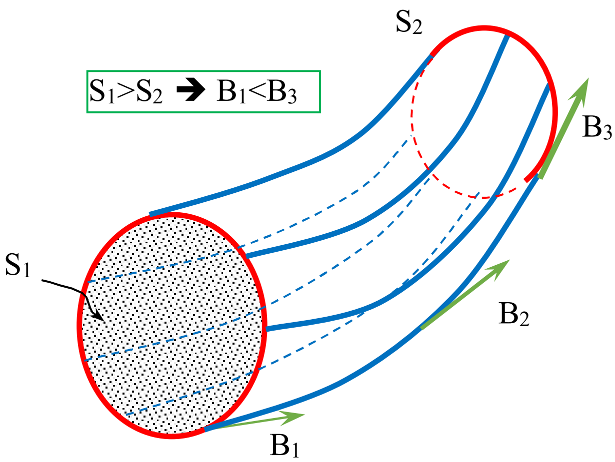

Figure 3-6: Flux tube segment

Through any cross-section of a flux tube passes the same magnetic flux φ. That is why the flux density varies with the size of the cross-section. If the cross-section if the flux tube were constant, the flux density B is constant. The magnetic field intensity H is constant only if the material is the same everywhere. Otherwise, it is inverse proportional to the permeability of the material.

3.3 Magnetic circuit

Consider the structure of Figure 3-7, made by ferromagnetic material with a constant permeability μ_{fe}.

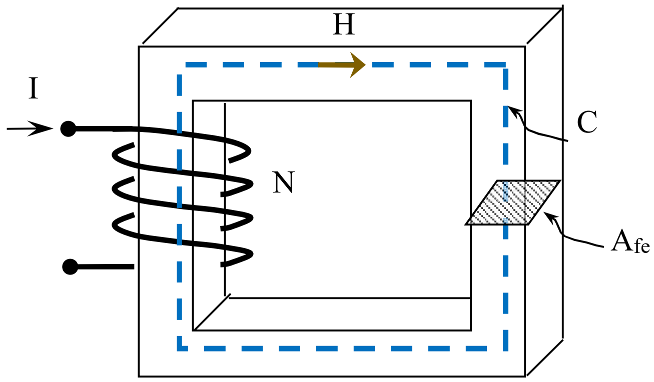

Figure 3-7: Ferromagnetic structure

Suppose that permeability μ would be much higher than the vacuum permeability, such way the magnetic flux is bounded into the ferromagnetic material.

Consider the line C, which length is "l_{fe}". Suppose that the cross-section A_{fe} is constant along the contour C.

The surface integral of the flux density B across the area A_{fe} (cross-section of the ferromagnetic structure), considering B constant in each point of the area (same assumption of the magnetic field intensity H) gives the expression of the magnetic flux.

$$\iint_S \overline{B} \cdot \overline{n} dA = BA_{fe} = \varphi$$

If you apply the law of Ampere to the contour C we have:

$$NI = \oint_C \overline{H} \cdot d\overline{l} = \oint_C Hdl = \oint_C \frac{B}{\mu_{fe}} dl = \oint_C \frac{\varphi}{A_{fe}\mu_{fe}} dl = \varphi \oint_C \frac{1}{A_{fe}\mu_{fe}} dl = \varphi \frac{l_{fe}}{A_{fe}\mu_{fe}} = \varphi \theta_{fe}$$

$$M = \frac{1}{\mu_{fe}} \frac{l_{fe}}{A_{fe}} \varphi = \theta_{fe} \varphi$$

where θ_{fe} is the reluctance of the magnetic path.

The inverse of the reluctance θ is called "permeance" Λ.

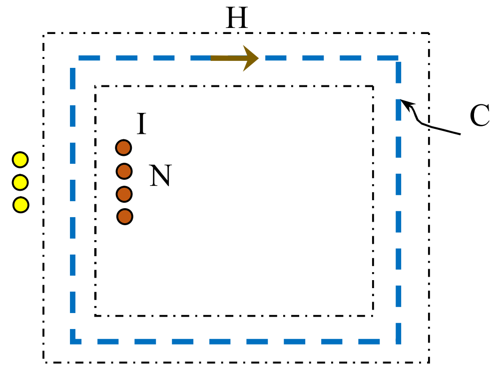

Figure 3-8: Application of the Ampere's law

Consider, now, a different configuration, in which the ferromagnetic structure is opened so to have an air-gap.

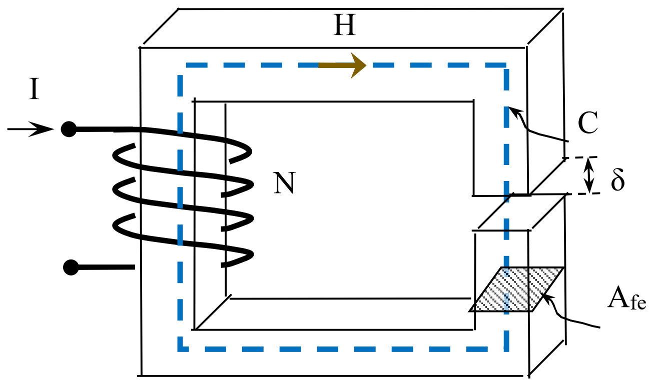

Figure 3-9: Ferromagnetic structure with air-gap

If the length of the air-gap is negligible respect to the dimensions of the ferromagnetic structure, the fringing effect (or side effect) (*effetti di bordo*) may be neglected, so to consider the cross-section A_δ of the magnetic flux into the air-gap equal to A_{fe}.

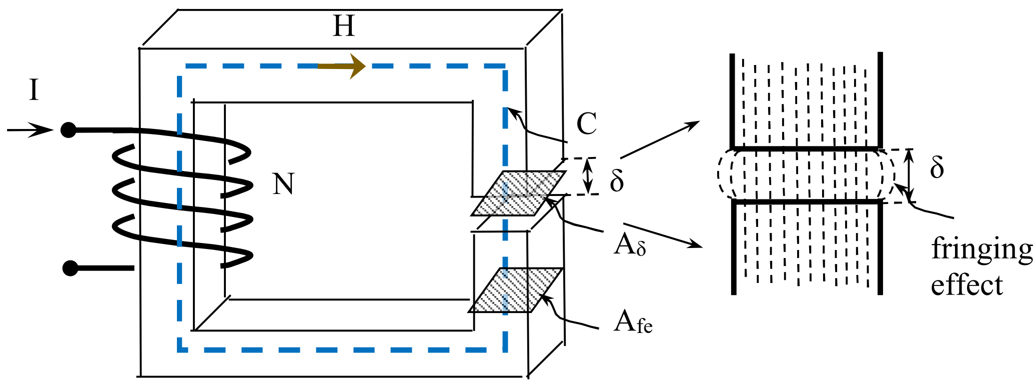

Figure 3-10: Ferromagnetic structure with air-gap (fringing effect)

Due to the fact that the magnetic flux φ is the same everywhere (both in the ferromagnetic material and the airgap, $\varphi = B_\delta\, A_\delta = B_{fe}\, A_{fe}$), neglecting the fringing effect ($A_\delta = A_{fe}$) means that the magnetic flux density B assumes the same value in the ferromagnetic material and in the air-gap ($B_\delta = B_{fe}$).

Call l_{fe} the length of the flux path inside the ferromagnetic material (a little bit lower than the above value, due to the size of the air-gap) and δ the size of the air-gap; the Ampere's law says:

$$NI = \oint_C \overline{H} \cdot d\bar{l} = \oint_C H dl = \int_{l_{fe}} H_{fe} dl + \int_{\delta} H_{\delta} dl = \int_{l_{fe}} \frac{B_{fe}}{\mu_{fe}} dl + \int_{\delta} \frac{B_{\delta}}{\mu_0} dl = \int_{l_{fe}} \frac{\varphi}{A_{fe}\mu_{fe}} dl + \int_{\delta} \frac{\varphi}{A_{\delta}\mu_0} dl =$$

$$= \varphi\left(\frac{l_{fe}}{A_{fe}\mu_{fe}} + \frac{\delta}{A_{\delta}\mu_0}\right) = \varphi(\theta_{fe} + \theta_{\delta})$$

where $B_{\delta} \approx B_{fe}$ and $A_{\delta} \approx A_{fe}$.

Usually the ferromagnetic material has a permeability much higher than that of the air (the relative permeability of Iron is about 5000). It means that the magnetic field intensity H is much lower in the ferromagnetic material than in the air: $H_{\delta} = B_{\delta}/\mu_o$, $H_{fe} = B_{fe}/\mu_{fe} << H_{\delta}$. A usually adopted simplification, in presence of an air-gap, is to consider $H_{fe} \approx 0$.

This means also that the reluctance of the magnetic path inside the ferromagnetic material is negligible respect to the reluctance of the air-gap.

$$NI = M = H_{fe} \cdot l_{fe} + H_{\delta} \cdot \delta = \varphi \cdot \theta_{fe} + \varphi \cdot \theta_{\delta} = \varphi \cdot \theta_{tot} \approx \varphi \cdot \theta_{\delta}$$

$$\theta_{\delta} = \frac{1}{\mu_o}\frac{\delta}{A_{\delta}} = \frac{1}{\mu_o}\frac{\delta}{A_{fe}}$$

In summary, the relationship between the magnetomotive force and the magnetic flux is the reluctance, in a similar way of the electric circuits, where the magnetomotive force is similar to a voltage generator, the magnetic flux to the current and the reluctance to the resistance.

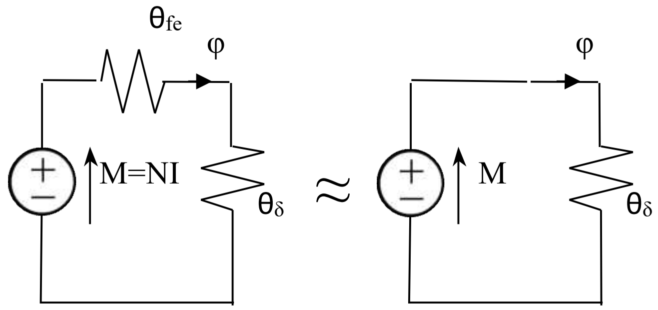

Figure 3-11: Magnetic circuit of the structure of Figure 3-9

3.4 Electromagnetic induction Law

Flux linkage (*flusso concatenato*) definition

$$\psi = N\varphi$$

Relationship between voltage and flux linkage Faraday/Lenz Law

$$v = e = \frac{d\psi}{dt}$$

"*e*" is called "induced voltage" or electromotive force (emf) (*forza elettromotrice, fem*)

Inductance relationship

$$\psi = Li$$

Inductance of a coil with ferromagnetic material

The inductance of the coil of Figure 3-12 has to be calculated by means of the following steps.

The magnetic flux φ which is linked with the coil is given by $\varphi = \frac{Ni}{\theta_{eq}} = \frac{Ni}{\theta_{fe} + \theta_\delta}$.

If the permeability of the ferromagnetic material is infinite, the magnetic flux is $\varphi = \frac{Ni}{\theta_\delta}$ where $\theta_\delta = \frac{1}{\mu_o}\frac{\delta}{A_{fe}}$

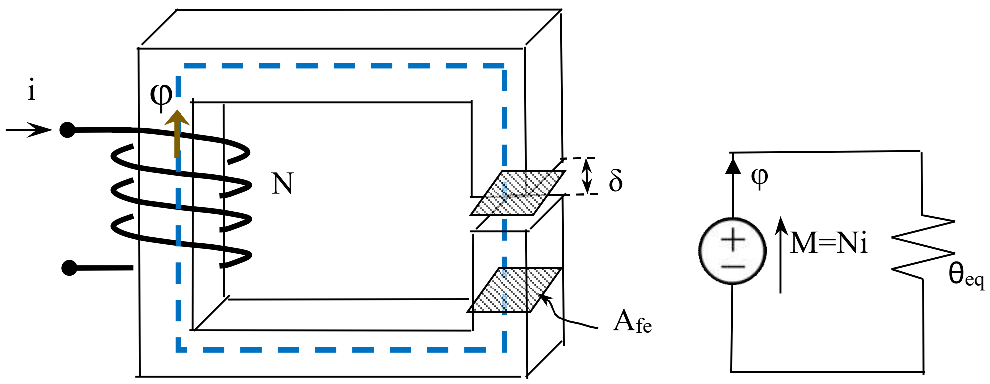

Figure 3-12: Inductance of a coil in a ferromagnetic structure with air-gap

The flux linkage, linked with the coil is $\psi = N\varphi$, so $\psi = N\frac{Ni}{\theta_{eq}} \Rightarrow L = \frac{N^2}{\theta_{eq}}$.

If there are more than one coil, the flux linked with a coil is produced by all the magnetomotive force of each coil. If the system is linear, you can apply the superposition principle: the stimulus is each current, the effect is the flux linkage of one coil. The effect on the flux linkage ψ_k of the coil "*k*" of the current i_k of the coil itself is represented by a constant coefficient L_{kk}, called self-inductance (*auto induttanza*). The effect on the flux linkage ψ_k of the coil "*k*" of the current i_j of the coil "*j*" different from "*k*" is represented by a constant coefficient L_{kj}, called mutual-inductance (*mutua induttanza*).

In general term, $L_{kj} = \left.\frac{\psi_k}{i_j}\right|_{i_{x \neq j} = 0}$: if j=k the L_{kj} represents a self-inductance otherwise the mutual inductance between coil j and coil k.

This means that the flux linked with the coil "*k*" in a ferromagnetic circuit with "*n*" coils is:

$$\psi_k = \sum_{i=1\cdots n} L_{ki} i_i$$

The effect of each magnetomotive force on the flux linkage may increase or decrease it. So you have to pay attention to the sign of each effect.

The voltage induced in the coil "*k*" is the derivative of the flux linkage, so (consider a linear system):

$$v_k = \frac{d\psi_k}{dt} = \sum_{i=1\cdots n} L_{ki}\frac{di_i}{dt}$$

A simple example is the circuit of Figure 3-13, with two coils.

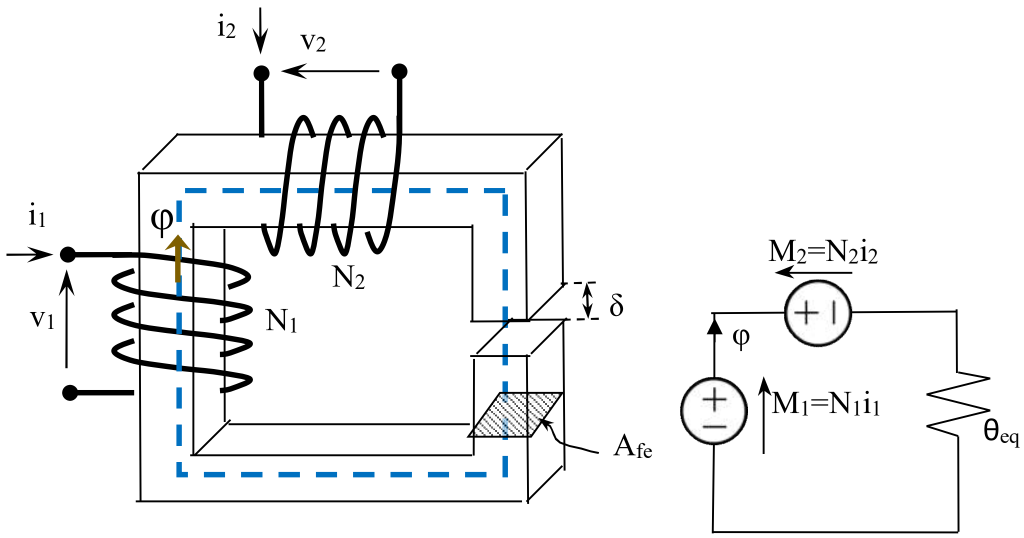

Figure 3-13: Ferromagnetic structure with air-gap and two coils

The magnetic flux due to the magnetomotive force M_1, linked with the coil 2 is opposite to the magnetic flux due to the magnetomotive force M_2, so the total flux linkage of coil 2 is given by the difference of the two effect:

$$\psi_2 = L_{22} i_2 - L_{21} i_1$$

The same

$$\psi_1 = L_{11} i_1 - L_{12} i_2$$

And the voltages

$$v_1 = \frac{d\psi_1}{dt} = L_{11}\frac{di_1}{dt} - L_{12}\frac{di_2}{dt}$$

$$v_2 = \frac{d\psi_2}{dt} = L_{22}\frac{di_2}{dt} - L_{21}\frac{di_1}{dt}$$

3.5 Energy

The energy is the integral of the instantaneous power.

For an inductance the instant power is

$$p = vi = \frac{id\psi}{dt} = Li\frac{di}{dt}$$

The energy is:

$$W = \int pdt = \int vidt = \int Lidi = \frac{1}{2}Li^2$$

For a two-coil mutual inductor we have that the total instantaneous power is the sum of the two powers. Consider the case in which the two effects are combined positively.

$$p_1 + p_2 = v_1 i_1 + v_2 i_2 = L_{11} i_1 \frac{di_1}{dt} + L_{12} i_1 \frac{di_2}{dt} + L_{22} i_2 \frac{di_2}{dt} + L_{21} i_2 \frac{di_1}{dt}$$

If the system is conservative, the energy is a state function and its value does not depend on the path you follow to reach the finale state of the system.

Suppose you have to calculate the variation of energy stored in the magnetic circuit, from a state of the system in which both the currents are equal to zero to a state characterized by two currents $i_1=I_1$ and $i_2=I_2$. Consider a path, varying i_1 from 0 to I_1 with $i_2=0$ and a second step varying i_2 from 0 to I_2 with $i_1=I_1$.

During the first path, the variation of the energy is ($i_2=0$):

$$W_{1st}^{\ I} = \int_{i_i=0,,i_2=0}^{i_i=I_1,,i_2=0} p_{tot}dt = \int_{i_i=0,,i_2=0}^{i_i=I_1,,i_2=0} L_{11}i_1 \frac{di_1}{dt} = \frac{1}{2}L_{11}I_1^{\ 2}$$

During the second path, the variation of the energy is ($i_1=I_1$):

$$W_{2nd}^{\ I} = \int_{i_i=I_1,,i_2=0}^{i_i=I_1,,i_2=I_2} p_{tot}dt = \int_{i_i=I_1,,i_2=0}^{i_i=I_1,,i_2=I_2} L_{12}I_1 \frac{di_2}{dt} + \int_{i_i=I_1,,i_2=0}^{i_i=I_1,,i_2=I_2} L_{22}i_2 \frac{di_2}{dt} = L_{12}I_1I_2 + \frac{1}{2}L_{22}I_2^{\ 2}$$

The total variation of the energy is

$$W_{tot}^{\ I} = W_{1st}^{\ I} + W_{2nd}^{\ I} = \frac{1}{2}L_{11}I_1^{\ 2} + L_{12}I_1I_2 + \frac{1}{2}L_{22}I_2^{\ 2}$$

If you choose a different path (varying i_2 from 0 to I_2 with $i_1=0$ and a second step varying i_1 from 0 to I_1 with $i_2=I_2$) you have

$$W_{tot}^{\ II} = W_{1st}^{\ II} + W_{2nd}^{\ II} = \frac{1}{2}L_{22}I_2^{\ 2} + L_{21}I_1I_2 + \frac{1}{2}L_{11}I_1^{\ 2}$$

But in a conservative system the energy does not depend on the path, so W_{tot}^{I} is equal to W_{tot}^{II}. A consequence is that the $L_{12}=L_{21}$.

In general term: $L_{kj}=L_{jk}$.

Call $L_{12}=L_{21}=L_m$.

The expression of the stored energy in a mutual inductor is (the sign before the mutual inductance depends on the combination of the two effects):

$$W_{tot} = \frac{1}{2}L_{11}I_1^{\ 2} \pm L_mI_1I_2 + \frac{1}{2}L_{22}I_2^{\ 2}$$

For three coils ($L_{12}=L_{21}$, $L_{13}=L_{31}$, $L_{32}=L_{23}$):

$$W_{tot} = \frac{1}{2}L_{11}I_1^{\ 2} + \frac{1}{2}L_{22}I_2^{\ 2} + \frac{1}{2}L_{33}I_3^{\ 2} \pm L_{12}I_1I_2 \pm L_{23}I_2I_3 \pm L_{31}I_3I_1$$

and so on.

3.6 Force

From the first Law of Thermodynamics, you have (no heat exchange):

$$dW = \delta L_e - \delta L_m$$

The meaning is: the variation of the energy stored in the device is the difference between the electrical work (incoming) and the mechanical work (outcoming).

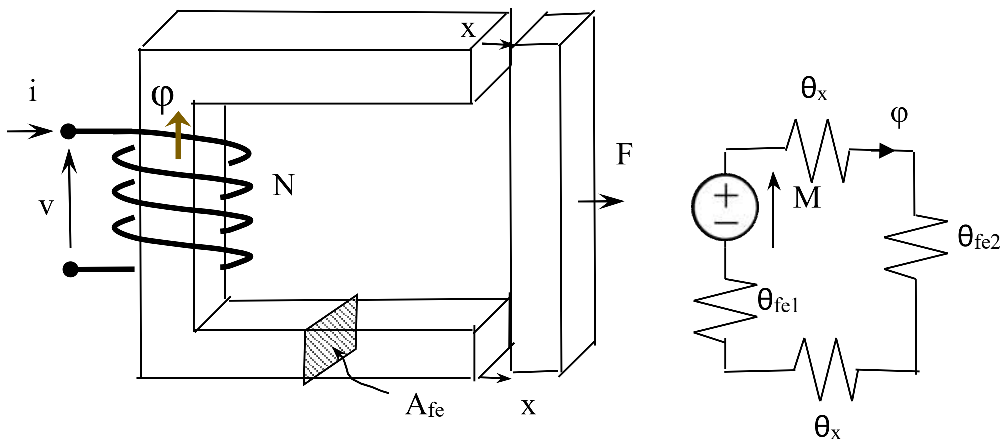

Figure 3-14: Relay

Consider the structure of Figure 3-14, with only one electrical port. Usually the electrical and the mechanical works are inexact differentials, while energy is a state function (exact differential). But for an ideal inductor the electric power is

$$p_e = vi = \frac{id\psi}{dt} \Rightarrow \delta L_e = id\psi$$

From the mechanical point of view

$$p_m = F\mathrm{v} = \frac{Fdx}{dt} \Rightarrow \delta L_m = Fdx$$

therefore

$$dW = id\psi - Fdx$$

It means that the flux linkage ψ and the position x are the state variables and $W=W(\psi,x)$. Then

$$dW = \frac{\partial W}{\partial \psi} d\psi - \frac{\partial W}{\partial x} dx$$

It means that

$$i = \left.\frac{\partial W}{\partial \psi}\right|_{x = const}$$

$$F = -\left.\frac{\partial W}{\partial x}\right|_{\psi = const}$$

In a linear system $\psi = Li$ (L is not a function of the current i, but of the only position x) and the energy stored in the magnetic circuit is

$$W = \frac{1}{2} Li^2 = \frac{1}{2} \frac{\psi^2}{L}$$

If $\mu_{fe}=\infty$ ($\rightarrow \theta_{fe}=0$), the inductance L is:

$$L = \frac{N^2}{\theta_{eq}} \Rightarrow \theta_{eq} = \theta_x + \theta_x = \frac{1}{\mu_o}\frac{2x}{A_{fe}} \Rightarrow W = \frac{1}{2}\frac{\psi^2}{N^2}\frac{1}{\mu_o}\frac{2x}{A_{fe}}$$

so

$$F = -\frac{\partial W}{\partial x}\bigg|_{\psi = const} = -\frac{1}{2}\frac{\psi^2}{N^2}\frac{1}{\mu_o}\frac{2}{A_{fe}}$$

This is the total force, given by two forces in correspondence of the air-gap. The force for each air-gap is:

$$F = -\frac{1}{2}\frac{\psi^2}{N^2}\frac{1}{\mu_o}\frac{1}{A_{fe}} = -\frac{1}{2}\varphi^2\frac{1}{\mu_o}\frac{1}{A_{fe}} = -\frac{1}{2}B^2 A_{fe}{}^2\frac{1}{\mu_o}\frac{1}{A_{fe}} = -\frac{1}{2}\frac{B}{\mu_o}BA_{fe} = -\frac{1}{2}HBA_{fe}$$

The force is attractive (negative sign) and the pressure σ is

$$\sigma = \left|\frac{F}{A_{fe}}\right| = \frac{1}{2}HB$$

which represents, also, the energy per unit volume, stored in the air-gap (in the ferromagnetic material, due the infinite value of the permeability, there is no stored energy).